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# GENERALIZED SUBHARMONIC RESPONSE OF A MISSILE WITH SLIGHT CONFIGURATIONAL ASYMMETRIES

by

Charles H. Murphy

June 1972



U.S. ARMY ABERDEEN RESEARCH AND DEVELOPMENT CENTER
BALLISTIC RESEARCH LABORATORIES
ABERDEEN PROVING GROUND, MARYLAND

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Charles H. Murphy

Exterior Ballistics Laboratory





RDT&E Project No. 1T061102A33D


A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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ABSTRACT

The quasilinear analysis of the angular motion of slightly asymmetric missiles with a cubic static moment shows the existence of nonharmonic steady-state solutions. These solutions are described by sums of three constant amplitude rotating angles. One angle rotates at the spin rate and the other two rotate in opposite directions at approximately one-third the spin rate. This generalized subharmonic motion can be very much larger than the usual harmonic trim motion and can occur for spin rates greater than eight times the resonant spin rate. The predictions of the quasilinear theory are compared with the results of exact numerical integration of the equations of motion with much better than 5% agreement in all cases.

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LIST OF ILLUSTRATIONS

## LIST OF SYMBOLS

| | |
|---|---|
| $b$ | $-\dot{\phi}_2/\dot{\phi}_1$ |
| $c_0$ , $c_2$ | cubic static moment coefficients |
| $C_{\tilde{m}}$ , $C_{\tilde{n}}$ | coefficients of the transverse components of the aerodynamic moment |
| $C_{M_0}$ | aerodynamic moment coefficient due to asymmetry |
| $C_{M_\alpha}$ | static moment coefficient |
| $C_{M_{\dot{\alpha}}}$ , $C_{M_q}$ | damping moment coefficients |
| $H$ | $-\rho \frac{S\ell^3}{2 I_y} (C_{M_q} + C_{M_{\dot{\alpha}}})$ |
| $I_x$ , $I_y$ | axial, transverse moments of inertia |
| $k_j$ | amplitude of the jth mode ($j = 1, 2$) |
| $k_3$ | amplitude of the harmonic trim motion |
| $k_{3m}$ | maximum value of $k_3$ |
| $\ell$ | reference length |
| $m$ | mass |

LIST OF SYMBOLS (CONTINUED)

| | |
|---|---|
| $m_a$ | $\dfrac{M_2 \delta_{T0}^2}{M_0}$ |
| $M_a$ | aerodynamic moment due to asymmetry |
| $M$ | $\rho \dfrac{S\ell^3}{2 I_y} C_{M_\alpha}$ |
| $M_0$ , $M_2$ | $\rho \dfrac{S\ell^3}{2 I_y} (c_0 , c_2)$ |
| $p$ | roll rate, $\dfrac{d\phi}{dt}$ |
| $P$ | $(I_x/I_y) \dfrac{p\ell}{V}$ , gyroscopic spin |
| $\tilde{q}$, $\tilde{r}$ | angular velocity components along $\tilde{Y}$, $\tilde{Z}$ axis |
| $s$ | dimensionless distance along flight path $\int_{t_0}^{t} V/\ell \, dt$ |
| $S$ | reference area |
| $t$ | time |
| $\tilde{v}$, $\tilde{w}$ | $\tilde{Y}$, $\tilde{Z}$ components of the velocity vector |
| $V$ | magnitude of the velocity vector |
| $X$, $\tilde{Y}$, $\tilde{Z}$ | nonrolling Cartesian coordinate axes |
| $\delta$ | $\lvert\tilde{\xi}\rvert$ , sine of the total angle of attack |
| $\delta_{T0}$ | $M_a/M_o$, trim angle for zero spin |

| | |
|---|---|
| $\tilde{\zeta}$ | $\dfrac{\tilde{\xi}}{\delta_{T0}}$ |
| $\lvert\tilde{\zeta}\rvert^2_{ej}$ | effective squared yaws (j = 1, 2, 3) |
| $\hat{\lambda}_j$ | damping rate of the j-modal amplitude (j = 1, 2) |
| $\tilde{\mu}$ | $\dfrac{(\tilde{q} + i\tilde{r})\ell}{V}$ |
| $\tilde{\xi}$ | $\dfrac{\tilde{v} + i\tilde{w}}{V}$ |
| $\rho$ | air density |
| $\tau$ | $(-M_0)^{\frac{1}{2}} s$ |
| $\phi$ | roll angle |
| $\dot{\phi}$ | roll rate, $(\frac{d\phi}{d\tau})$ |
| $\dot{\phi}_m$ | roll rate at which the maximum harmonic response occurs |
| $\phi_j$ | j-modal phase angle, (j = 1, 2) |
| $\phi_3$ | trim mode phase angle $(\phi + \phi_{30})$ |
| $\phi_r$ | phase angle between the fast mode and trim mode, $\phi_1 - \phi_3$ |
| $\phi_s$ | phase angle for the magnitude of the angular motion for symmetric missiles, $\phi_1 - \phi_2$ |
| $\Psi$ | $2\phi_1 - \phi_2 - \phi_3 = \phi_s + \phi_r$ |

LIST OF SYMBOLS (CONTINUED)

Superscripts

| | |
|---|---|
| $\tilde{(\ )}$ | components in a nonrolling coordinate system |
| $(\ )'$ | derivatives with respect to s, the arc length in calibers |
| $\dot{(\ )}$ | $\frac{d}{d\tau}(\ )$ |
| $\hat{(\ )}$ | $(\ )(-M_0)^{-\frac{1}{2}}$ |
| $\bar{(\ )}$ | complex conjugate |

Subscripts

| | |
|---|---|
| 0 | evaluated at $\tau = 0$ |
| av | average over a distance that is large with respect to the wavelength of either $\phi_r$ or $\phi_s$ |

# I. INTRODUCTION

The theory of the linear motion of a "slightly asymmetric" missile was first developed by Nicolaides.[1*] The assumed aerodynamic force and moment have the same form for an asymmetric missile as for a symmetric missile with the exception of the presence in the asymmetric case of constant amplitude force and moment terms whose orientations are fixed relative to the missile. These terms induce a trim angle of attack that rolls with the missile (tricyclic motion). For a constant roll rate, the amplitude of this trim angle of attack is a function of the roll rate with a maximum angle occurring when the roll rate equals the natural pitch frequency (resonance). Recently, asymptotic relations have been derived for trim motion when spin varies through a constant natural frequency[2] and when the natural frequency varies through equality with a constant spin rate.[3]

In 1959 Nicolaides[4] extended his analysis of asymmetric missiles to include nonlinear roll-orientation-dependent terms and thereby introduced the concepts of "spin lock-in" and "catastrophic yaw." Since then, spin lock-in has been studied by a number of authors.[5-6] The angular trim produced by a nonlinear static moment has been treated by Kanno.[7] The quasilinear analysis technique, which has been most successful in describing the angular motion of symmetric missiles,[8-9] has recently been extended to the general angular motion of a slightly asymmetric missile.[10] The detailed nonlinear behavior near resonance has been discussed by Clare.[11] Finally, Nayfeh and Saric[12] have shown that the results of References 10-11 can be obtained by the more sophisticated method of multiple scales.

The complex differential equation for the two-dimensional angular motion of a slightly asymmetric missile with a cubic static moment is quite similar in form to Duffing's equation.[13-14] The harmonic response curve has the same form as for the one-dimensional Duffing's equation. The response curve is a multivalued function of spin with

*References are listed on page 39.

an in-phase solution and two out-of-phase solutions existing near resonance, and it can be shown that the intermediate size solution is unstable.[7] Since the harmonic response for the two-dimensional motion is a <u>constant amplitude</u> coning motion, the mathematical analysis is much simpler than for Duffing's equation and does not require any approximations.

As a result of this similarity with Duffing's equation, the question of the existence of subharmonic solutions naturally arises. In the one-dimensional case, these are obtained by requiring the quasilinear transient frequency to be one-third the forcing function frequency. It is then shown that for sufficiently small damping, subharmonic solutions exist for spin rates three times the resonance spin rate.

The situation is somewhat more complicated for the two-dimensional case. There are two transient frequencies present and, therefore, the condition for subharmonic response is not clear. In this paper we will obtain a condition for subharmonic response and derive the equations for the amplitudes of the three modes of angular motion. These theoretical predictions will then be compared with the results of numerical integration of the fourth order equations of angular motion.

## II. NONLINEAR ANALYSIS

Although a missile-fixed coordinate system is very commonly used for the analysis of the flight of missiles and aircraft, a related nonspinning coordinate system which pitches and yaws with the missile is most useful for treating the motion of symmetric missiles. For this coordinate system with axes $X, \tilde{Y}, \tilde{Z}$, the X-axis lies along the axis of symmetry of the missile and the $\tilde{Y}$- and $\tilde{Z}$-axes are so constrained that the $\tilde{Y}$-axis is initially in the horizontal plane and the angular velocity of the coordinate system has a zero X-component. The angular motion of the symmetric missile can be described by a complex angle of attack, $\tilde{\xi}$, and a complex angular velocity, $\tilde{\mu}$, where

$$\tilde{\xi} = (\tilde{v} + i\,\tilde{w})/V \tag{1}$$

$$\tilde{\mu} = (\tilde{q} + i\,\tilde{r})\ell/V \tag{2}$$

$\tilde{\xi}$ is a complex number whose magnitude is the sine of the angle between the velocity vector and the missile's axis of symmetry (the total angle of attack) and whose orientation locates the plane of this angle.

By definition, a slightly asymmetric missile has a transverse moment expansion of the same form as for a symmetric missile with an added constant amplitude term that rotates with the missile. This type of term can be introduced by a small control surface deflection or a small manufacturing inaccuracy. If we limit the nonlinearities under consideration to a cubic static moment and neglect any Magnus moment contribution, the transverse moment expansion for a slightly asymmetric missile assumes the form

$$\begin{aligned} C_{\tilde{m}} + i\,C_{\tilde{n}} = &- i\,C_{M_0}\,e^{i\phi} - i(c_0 + c_2\,\delta^2)\tilde{\xi} \\ &- i\,C_{M_{\dot{\alpha}}}\,\tilde{\xi}' + C_{M_q}\,\tilde{\mu} \end{aligned} \tag{3}$$

where

$$\delta^2 = |\tilde{\xi}|^2 = \tilde{\xi}\,\bar{\tilde{\xi}}$$

$$\phi' = p\ell/V$$

For this moment the differential equation* of motion is[11]

---

**The effects of lift and drag have been neglected for simplicity. These effects plus that of a linear Magnus moment can be easily added. Small geometric angles ($\delta < 0.2$) are also assumed so that certain geometric nonlinearities can be omitted.*

$$\tilde{\xi}'' + (H - iP)\tilde{\xi}' - (M_0 + M_2\ \delta^2)\tilde{\xi} = - M_a e^{i\phi} \tag{4}$$

where H, P, $M_0$, $M_2$ and $M_a$ are defined in the list of symbols. For constant spin, the asymmetric moment forcing function induces an aerodynamic trim angle that rotates with the missile. If both the spin and the cubic static moment terms vanish ($P = M_2 = 0$), this trim angle becomes the linear trim angle for zero spin and is given by the relation

$$\delta_{T0} = M_a/M_0 \tag{5}$$

$\tilde{\xi}$ can now be scaled with respect to this zero-spin trim angle by the substitution

$$\tilde{\zeta} = \tilde{\xi}/\delta_{T0} \tag{6}$$

A further simplification is possible by use of a new independent variable $\tau$, where

$$\tau = (-M_0)^{\frac{1}{2}}\ s \tag{7}$$

These changes of variable reduce Equation (4) to a much simpler form:

$$\ddot{\tilde{\zeta}} + (\hat{H} - i\hat{P})\dot{\tilde{\zeta}} + (1 + m_a|\tilde{\zeta}|^2)\tilde{\zeta} = e^{i\phi} \tag{8}$$

where

$$(\hat{\ }) = (\ )\ (-M_0)^{-\frac{1}{2}}$$

$$m_a = M_2\,\delta_{T0}^2/M_0$$

$$(\dot{\ }) = \frac{d}{d\tau}(\ )$$

$m_a$ is the ratio of the cubic part of the static moment to the linear part when the angle of attack is equal to the linear zero-spin trim angle. Thus it is a measure of the nonlinearity as well as of the amplitude of the asymmetric moment.

Duffing's equation for one-dimensional motion could be written in the form

$$\ddot{x} + h\dot{x} + (1 + ax^2)x = \cos \dot{\phi}\,\tau \qquad (9)$$

For constant roll rate, small ratio of the moments of inertia ($\hat{P} \doteq 0$) and planar motion, the left side of Equation (8) takes on the form of Duffing's equation but the forcing function will not reduce to Duffing's forcing function. Indeed, for this circular forcing function, planar motion is impossible. If we take the real part of Equation (8), the forcing function has the same form as Duffing's but the nonlinear term contains the imaginary component of the angle as well as the real component. Thus Equation (4) is similar to Duffing's equation but is not a generalized form of it.

The harmonic response solution of Equation (8) for constant roll rate can be easily obtained by assuming a solution of the form

$$\tilde{\zeta} = k_3 e^{i\phi_3}, \qquad \phi_3 = \phi + \phi_{30} \qquad (10)$$

where $k_3$ and $\phi_{30}$ are constants.

Direct substitution of Equation (10) in Equation (8) yields:

$$k_3 e^{i\phi_{30}} = [1 - (1 - I_x/I_y)\dot{\phi}^2 + m_a k_3^2 + i\hat{H}\dot{\phi}]^{-1} \qquad (11)$$

This cubic equation in $k_3$ is plotted in Figure 1 for both positive and negative $m_a$. These curves and the instability[7] of the intermediate size solution are very similar to corresponding features of Duffing's equation. The maximum value of the harmonic response, $k_{3m}$, and the spin rate at which it occurs, $\dot{\phi}_m$, can be easily estimated from Equation (11) by letting $\phi_{30}$ be $-\pi/2$ and assuming the moment of inertia ratio to be small.

$$\dot{\phi}_m^2 = (1/2)\ \{1 + [1 + 4m_a\hat{H}^{-2}]^{\frac{1}{2}}\ \} \doteq m_a^{\frac{1}{2}}/\hat{H} \qquad (12)$$

$$k_{3m} = [\hat{H}\ \dot{\phi}_m]^{-1} \doteq m_a^{-1/4}\ \hat{H}^{-\frac{1}{2}} \qquad (13)$$

The approximate forms of Equations (12-13) are, of course, valid only when $m_a\hat{H}^{-2}$ is large.

The form of the solution to the linear version of Equation (8) is the usual tricyclic equation

$$\tilde{\zeta} = k_1 e^{i\phi_1} + k_2 e^{i\phi_2} + k_3 e^{i\phi_3} \qquad (14)$$

where

$$\dot{k}_j = \hat{\lambda}_j\ k_j \qquad j = 1,\ 2$$

The first two terms in Equation (14) are modes of free oscillations induced by initial conditions and for constant $\hat{\lambda}_j$ their amplitudes are exponentially damped.

These free oscillation modes are so numbered that $|\dot{\phi}_1| \geq |\dot{\phi}_2|$ . For a statically stable missile, the larger frequency, which is commonly called the nutational frequency by ballisticians, has the same sign as the spin while the smaller frequency (the ballistician's precessional frequency) has a sign opposite to that of the spin. If the actual solution to the nonlinear Equation (8) can be approximated by a solution with the form of Equation (14), the nonlinear coefficient of $\tilde{\zeta}$ can be approximated by

$$|\tilde{\zeta}|^2 = k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 \cos\phi_s \tag{15}$$

$$+ 2\, k_1k_3 \cos\phi_r + 2\, k_2k_3 \cos\,(\phi_s - \phi_r)$$

where

$$\phi_s = \phi_1 - \phi_2 \text{ and } \phi_r = \phi_1 - \phi_3$$

$\phi_s$ is the phase angle for the magnitude of the angular motion for a symmetric missile and is constant for a gyroscopic stability factor of unity. $\phi_r$ is the phase angle between the fast (nutational) mode and the trim mode and is constant for resonance ($\dot{\phi}_1 = \dot{\phi}$). The parameters ($\hat{\lambda}_j$, $\dot{\phi}_j$, $k_3$, $\phi_{30}$) of Equation (14) can be obtained as functions of $k_j$ by an averaging process. If the spin rate is not near zero or $\dot{\phi}_1$ these relations take the form:[10]

$$\hat{\lambda}_1 = - [\hat{H}\dot{\phi}_1 + \ddot{\phi}_1 - 2m_a k_2 k_3 \,(\sin \Psi)_{av}][2\dot{\phi}_1 - \hat{P}]^{-1} \tag{16}$$

$$\hat{\lambda}_2 = - [\hat{H}\dot{\phi}_2 + \ddot{\phi}_2 + m_a\, k_1^2\, k_3\, k_2^{-1}\, (\sin \Psi)_{av}][2\dot{\phi}_2 - \hat{P}]^{-1} \tag{17}$$

$$\dot{\phi}_j(\dot{\phi}_j - \hat{P}) = 1 + m_a |\tilde{\zeta}|^2_{ej} \qquad j = 1, 2 \tag{18}$$

$$k_3 \, [1 - (1 - I_x/I_y)\dot{\phi}^2 + m_a|\tilde{\zeta}|^2_{e3}] = \cos \phi_{30} \tag{19}$$

$$\hat{H}\dot{\phi} \, k_3 + m_a \, k_1^2 k_2 \, [\sin \Psi]_{av} = - \sin \phi_{30} \tag{20}$$

where

$$\Psi = \phi_s + \phi_r = 2\phi_1 - \phi_2 - \phi_3$$

$$|\tilde{\zeta}|^2_{e1} = k_1^2 + 2 \, k_2^2 + 2 \, k_3^2 + 2 \, k_2 k_3 \, [\cos \Psi]_{av}$$

$$|\tilde{\zeta}|^2_{e2} = k_2^2 + 2 \, k_1^2 + 2 \, k_3^2 + k_1^2 \, k_3 \, k_2^{-1} \, [\cos \Psi]_{av}$$

$$|\tilde{\zeta}|^2_{e3} = k_3^2 + 2 \, k_1^2 + 2 \, k_2^2 + k_1^2 \, k_2 \, k_3^{-1} \, [\cos \Psi]_{av}$$

## III. STEADY-STATE NONHARMONIC MOTION

For a dynamically stable missile the transient motion induced by initial conditions damps out and usually only the harmonic trim motion represented by $k_3$ remains. In the case of Duffing's equation, another steady-state solution is possible under certain conditions. This solution is a subharmonic one with a frequency of $\dot{\phi}/3$. As we shall see, Equation (4) can have a somewhat more complicated steady-state solution which is a generalized subharmonic response.

For a steady-state motion, $\ddot{\phi}_j$ is zero and Equation (14) will yield a nonharmonic steady motion for three combinations of $k_j$ and $\hat{\lambda}_j$:

(1) $k_1 = 0, \quad \hat{\lambda}_2 = 0$

(2) $k_2 = 0, \quad \hat{\lambda}_1 = 0$

or (3) $\hat{\lambda}_1 = \hat{\lambda}_2 = 0$. A quick inspection of Equations (16-17) shows that the third case is the only possibility and then only if the average of $\sin\Psi$ is nonzero. A condition for generalized subharmonic response is, therefore,

$$\dot{\Psi} = 2\,\dot{\phi}_1 - \dot{\phi}_2 - \dot{\phi} = 0 \tag{21}$$

Equations (16-17) for zero damping now provide a condition for $\Psi$ and a relation between the modal amplitudes

$$\sin\Psi = \hat{H}\,\dot{\phi}_1\,(2m_a\,k_2\,k_3)^{-1} \tag{22}$$

$$k_1^2 = 2\,b\,k_2^2 \tag{23}$$

where $b = -\dot{\phi}_2/\dot{\phi}_1$

Equations (18-23) now provide a set of seven equations in seven unknowns, $(k_1, k_2, k_3, \dot{\phi}_1, \dot{\phi}_2, \phi_{30}, \Psi)$.

Equations (18-20) can be greatly simplified if the ratio of moments of inertia is neglected and Equation (23) used to eliminate $k_1$ . The results of this simplification are given in Table I. Under the further simplification of no aerodynamic damping ($\hat{H} = 0$), Equation (22) reduces to $\Psi = 0, \pi$ and the fourth equation of Table I requires $\phi_{30} = 0, \pi$ . For the four possible choices of $\phi_{30}$ and $\Psi$ , solutions of Equation (21) and the remaining three equations of Table I were sought. The results are plotted in Figures 2-3. These figures give $k_2$ and $k_3$ as functions of $\dot{\phi}$ for various values of $m_a$ . Two solution sets of $k_2$ and $k_3$ are shown. In Figure 2, the two sets of values of $k_2$ are separated by plotting one as positive up and the second as positive down. In Figure 3, $k_3$ goes through zero for one set of solutions. At this point, say at $\dot{\phi} = A$, $\phi_{30}$ jumps from $\pi$ to 0 and this is shown by plotting $k_3$ as positive down after the shift in phase. The solutions identified by the dashed line have the values $\Psi = 0$, $\phi_{30} = \pi$ for all values of $\dot{\phi}$ while the solutions identified by a solid line have the values $\Psi = \pi$ , $\phi_{30} = \pi$ for $\dot{\phi}$ less than A and the values $\Psi = 0$, $\phi_{30} = 0$ for $\dot{\phi}$ greater than A.

As a result of numerical integrations described later in this report, it is conjectured that the dashed solution is unstable and only the solution identified by the solid curves can be realized by an actual missile in flight.

As can be seen from Figure 2, generalized subharmonics exist when $\dot{\phi} < 3$ for negative $m_a$ and when $\dot{\phi} > 3$ for positive $m_a$ . The precise value for the appearance of a subharmonic can be computed by setting $k_2$ equal to zero in the frequencies as given by Table I and inserting the result in Equation (21):

$$\dot{\phi} = 3\ [1 + 2\ m_a\ k_3^2]^{\frac{1}{2}} \tag{24}$$

Since the amplitude of a pure harmonic, $k_3$, at $\dot{\phi} = 3$ is quite close to 1/8, we see that subharmonics appear slightly to the left of $\dot{\phi} = 3$ for negative $m_a$ and slightly to the right of this point for positive $m_a$.

Table I. Frequencies and Harmonic Response for $I_x/I_y = 0$ and $k_1^2 = 2\ b\ k_2^2$

$$\dot{\phi}_1 = \{1 + m_a[2\ (b + 1)\ k_2^2 + 2\ k_3^2 + 2\ k_2\ k_3 \cos \Psi]\}^{\frac{1}{2}}$$

$$\dot{\phi}_2 = -\ \{1 + m_a[(4\ b + 1)\ k_2^2 + 2\ k_3^2 + 2\ b\ k_2\ k_3 \cos \Psi]\}^{\frac{1}{2}}$$

$$k_3\ \{1 - \dot{\phi}^2 + m_a\ [2\ (2\ b + 1)\ k_2^2 + k_3^2 + 2\ b\ k_2^3\ k_3^{-1} \cos \Psi]\} = \cos \phi_{30}$$

$$\hat{H}\ (\dot{\phi}\ k_3^2 - \dot{\phi}_2\ k_2^2)/k_3 = -\ \sin \phi_{30}$$

## IV. APPROXIMATE RELATIONS

A simple approximate relation for $k_2$ can be obtained by neglecting $k_3$ in the frequency equations of Table I and approximating b by 1. Figure 3 shows that $k_3$ is less than .15 for the region of interest and the relations for the frequencies show the b approximation to be good. If we make these approximations and substitute the frequency equations from Table I in Equation (21), a quadratic equation in $k_2^2$ results. One of the roots vanishes near $\dot{\phi} = 3$. If we approximate the radical in the relation for this root, a very simple expression results

$$m_a \, k_2^2 = \frac{(\dot{\phi}^2 - 9)(\dot{\phi}^2 - 1)}{6(7\,\dot{\phi}^2 - 11)} \tag{25}$$

Equation (25) matches the exact curves of Figure 2 over the range $2 < \dot{\phi} < 10$ with an accuracy of $\pm .02\, m_a^{-\frac{1}{2}}$.

In the vicinity of $\dot{\phi} = 3$, Equation (25) can be simplified to

$$m_a \, k_2^2 = 2\,(\dot{\phi} - 3)/13 \tag{26}$$

The corresponding approximations for the frequencies are

$$\dot{\phi}_1 = 1 + 4\,(\dot{\phi} - 3)/13 \tag{27}$$

$$\dot{\phi}_2 = -\,[1 + 5\,(\dot{\phi} - 3)/13] \tag{28}$$

If Equations (27-28) predicted the true subharmonic frequency present for Duffing's equation, i.e. $\dot{\phi}/3$ , they would have the form

$$|\dot{\phi}_j| = 1 + (\dot{\phi} - 3)/3 \tag{29}$$

The similarity between Equations (27-28) and Equation (29) leads us to call the results of this paper generalized subharmonic response.

## V. EFFECT OF DAMPING

In order for subharmonic motion to be observable in practice, some damping must be present to eliminate the transient motion. Equation (22) indicates an upper bound on $\hat{H}$ since the sine cannot exceed one. As $\hat{H}$ is increased from zero, the two pairs of solutions for $k_2$, $k_3$ approach each other and coalesce for the maximum value of $\hat{H}$ .

In Figure 4, the maximum value of $\hat{H}$ is plotted versus $\dot{\phi}$ for $m_a = .2$. Since $\hat{H} = .01$ induces an amplitude decrement of 3% per cycle of the linear frequency, the restriction on damping is seen to be quite severe. The corresponding curves for $k_2$ and $k_3$ are compared with those for the solid lines (stable solution) in Figure 5. We see that damping has little effect on $k_2$ although it does smooth out the variation in $k_3$. The jump in phase angle $\phi_{30}$ as $k_3$ goes through zero is replaced by a continuous variation in phase.

## VI. COMPUTED MOTION

The theory of this report gives us sufficient information to verify the existence of generalized subharmonic motion. For the values of $\hat{H}$, $m_a$ and $\dot{\phi}$ given in Table II, values of $\Psi$, $k_1$, $k_2$, $k_3$, $\phi_1$, $\phi_2$ and $\phi_{30}$ can be computed; these values are also listed in Table II. All of the parameters of the tricyclic motion described by Equation (14) are available except for $\phi_{10}$ and $\phi_{20}$, although the

Table II. Parameters for Sample Subharmonic Motion

| | | |
|---|---|---|
| $m_a$ = 1.0 ; | $\dot{\phi}$ = 3.5 ; | $\hat{H}$ = .0230 |
| $\Psi$ | 2.6180 (150°) | .5631 (32.26°) |
| $k_1$ | .4281 | .3745 |
| $k_2$ | .2976 | .2613 |
| $k_3$ | .0890 | .0952 |
| $\dot{\phi}_1$ | 1.1534 | 1.1563 |
| $\dot{\phi}_2$ | - 1.1931 | - 1.1875 |
| $\phi_{10}$ | .1000 (5.73°) | .1000 (5.73°) |
| $\phi_{20}$ | .6892 (39.49°) | 2.7513 (157.64°) |
| $\phi_{30}$ | 3.1760 (181.97°) | 3.1688 (181.56°) |
| $\tilde{\zeta}_{H_0}$ | .5667 | .0358 |
| $\tilde{\zeta}_{V_0}$ | .2289 | .1342 |
| $\dot{\tilde{\zeta}}_{H_0}$ | .1872 | .0839 |
| $\dot{\tilde{\zeta}}_{V_0}$ | - .0941 | .3846 |

values of $\Psi$ and $\phi_{30}$ specify $2\,\phi_{10} - \phi_{20}$ . A simple choice of $\phi_{10} = .1$ allows us to compute a set of initial values of $\tilde{\zeta}$ and $\dot{\tilde{\zeta}}$ for the generalized subharmonic motion.

Equation (8) was numerically integrated using both sets of initial conditions. No steady-state motion was found for the set of initial conditions associated with $\Psi = 32°$ , but a steady-state motion for the other set of initial conditions was observed and is shown in Figure 6. In all of our other numerical integrations, the solutions associated with the smaller $\Psi$ were never observed although most of those associated with the larger $\Psi$ have been. It is, therefore, our conjecture that the motions associated with the smaller $\Psi$'s are unstable.

The motion shown in Figure 6 appears in a much simpler form if we transform Equation (14) to missile-fixed coordinates by multiplying by $\exp(-\,i\,\phi)$ and make use of Equation (23) and the definition of $\Psi$:

$$\tilde{\zeta} = \{k_2[(2\,b)^{\frac{1}{2}}\,e^{i\phi_r} + e^{i(2\,\phi_r - \Psi)}] + k_3\}\,e^{i\phi_{30}} \tag{30}$$

The two terms involving $\phi_r$ represent an epicyclic motion with a nutational frequency that is twice the precessional frequency and a nutational amplitude that is approximately 40% of the maximum of the motion. The maximum value of the epicycle occurs for $\phi_r = \Psi$ . This epicyclic motion appears in the catalogue of such motions on page 63 of Reference 16 and is a closed curve with a single loop. The points of the curve plotted in Figure 6 were transformed to the non-rotating coordinate system and plotted in Figure 7. As can be seen, they fall precisely on a closed curve with a single loop.

The result of the exact numerical integration of Equation (8) as given by Figure 6 has been fitted by the usual tricyclic data analysis[1,15] used for ballistic range tests and the various tricycle parameters compared with the quasilinear prediction of Table II. The agreement is quite good. This process has been repeated for values of $\dot{\phi}$ up to 10 with equally good results, which are given in Figure 5. Thus the predicted generalized subharmonic response does occur for quite large values of $\dot{\phi}$ . Since it is not possible in the numerical work to use the maximum values of $\hat{H}$, lesser values were used as indicated. In Figure 8 the results for $k_2$ and $k_3$ are compared with their predictions for the appropriate values of $\hat{H}$ and it can be seen that in most cases the agreement is better than 5% .

## VII. DISCUSSION

In Figure 5 the harmonic response ($k_2 = 0$) is shown as a function of spin rate. At a spin rate of 9 the harmonic response is .012 and the harmonic component of the subharmonic motion is .14. The two generalized subharmonic amplitudes, however, are much larger ($k_1 = 4.2$, $k_2 = 3.0$) and steady-state motion of maximum amplitude 7.2 can occur. This steady-state motion is <u>600 times</u> the harmonic steady-state motion. If it occurs, it would have an important unexpected effect of the flight of the missile.

It should be noted that there are rather severe limitations on the occurrence of special steady-state motion at high spin rates ($\dot{\phi} > 3$). They are:

1. small aerodynamic damping, $\hat{H}$ (less than 6% decrement per pitch cycle);

2. nonlinear static moment which becomes more stable at larger angles ($c_2 < 0$);

3. initial conditions near the generalized subharmonic motion.

Thus generalized subharmonic motion should trouble the missile designer very infrequently. When it does occur, it can have a very large and puzzling effect since it occurs for spin rates which most designers consider to be very safe.


## ACKNOWLEDGMENT

The author is indebted to Professor I. D. Jacobson of the University of Virginia for several stimulating discussions which initiated the work of this report.

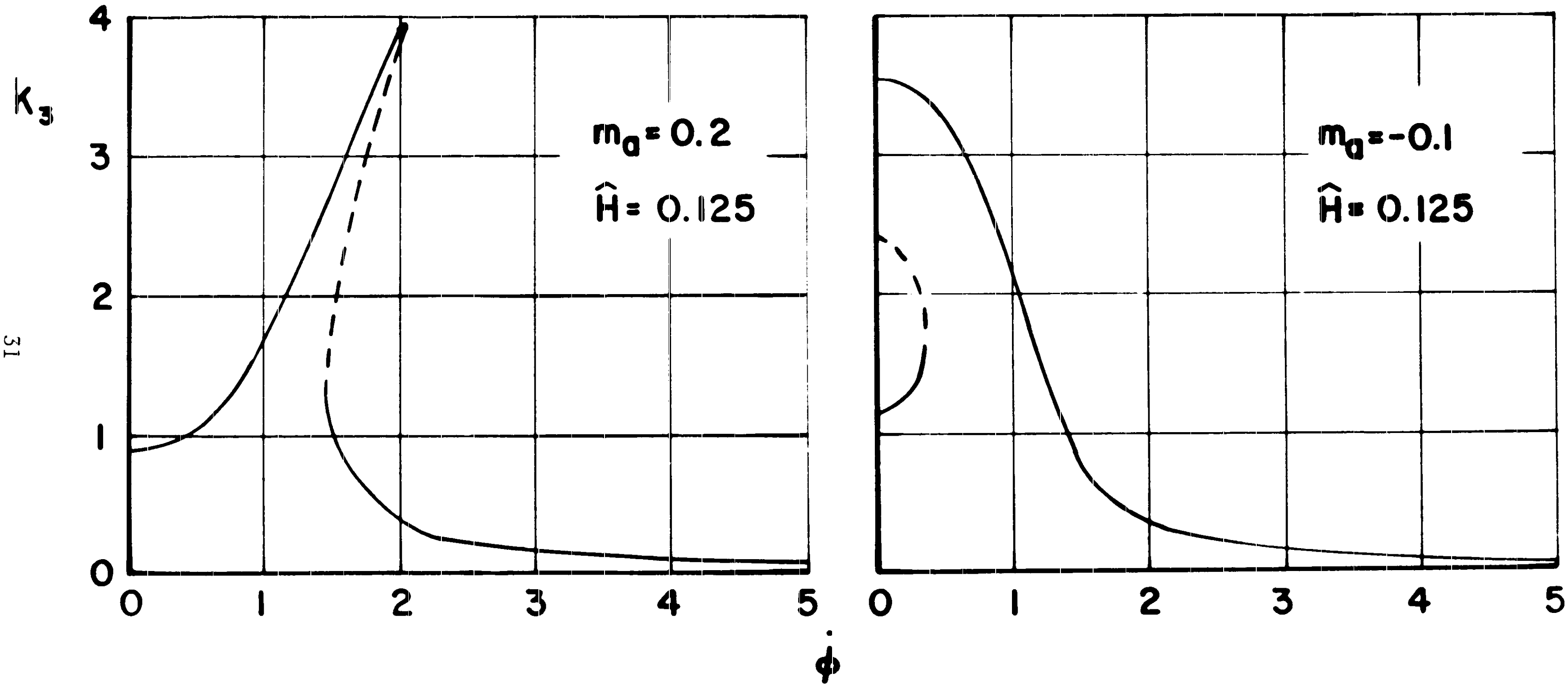


Figure 1. Nonlinear Harmonic Response

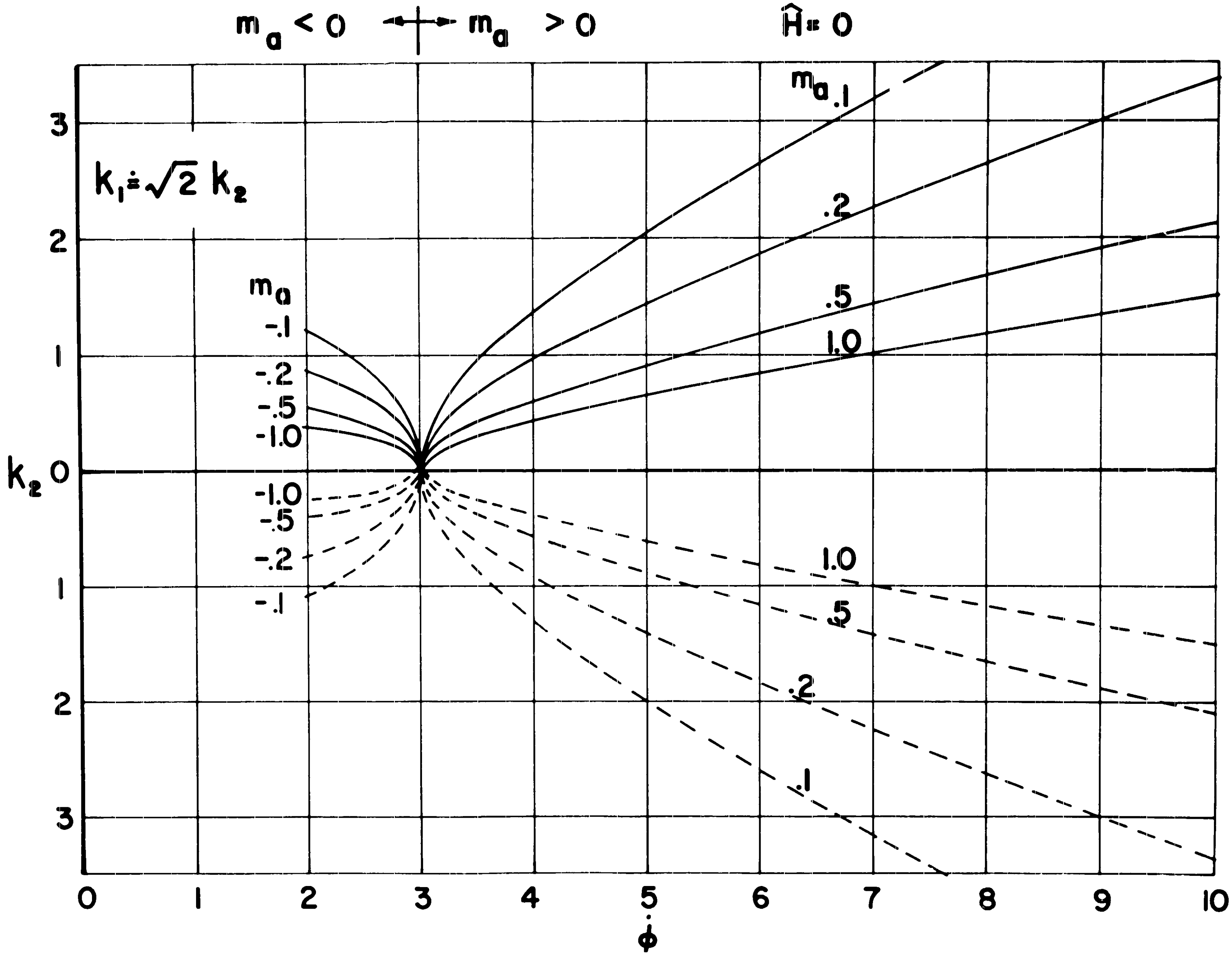


Figure 2. $K_2$ versus $\dot{\phi}$ for $|m_a| \leq 1$ and No Aerodynamic Damping

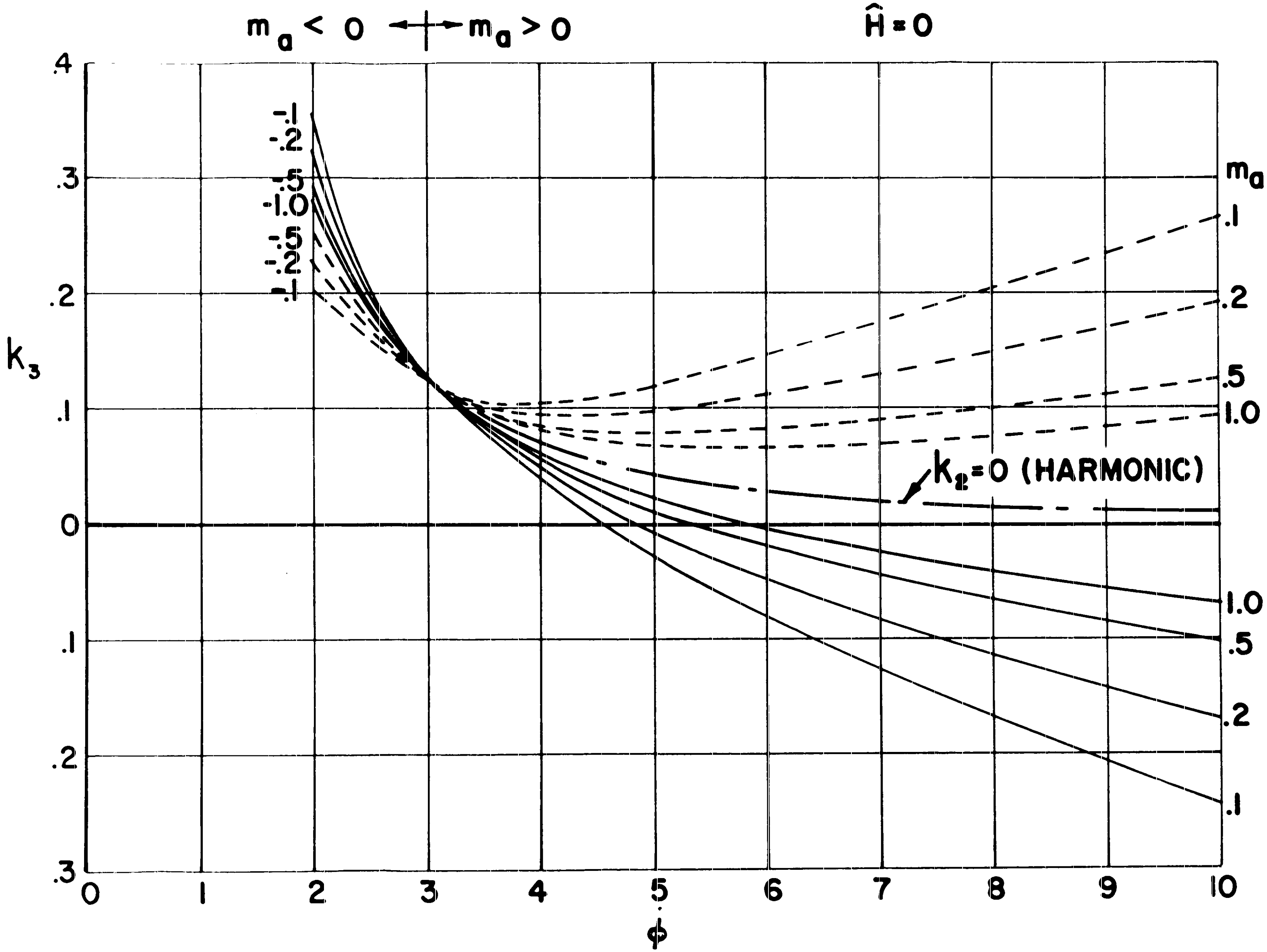


Figure 3. $K_3$ versus $\dot{\phi}$ for $|m_a| \leq 1$ and No Aerodynamic Damping

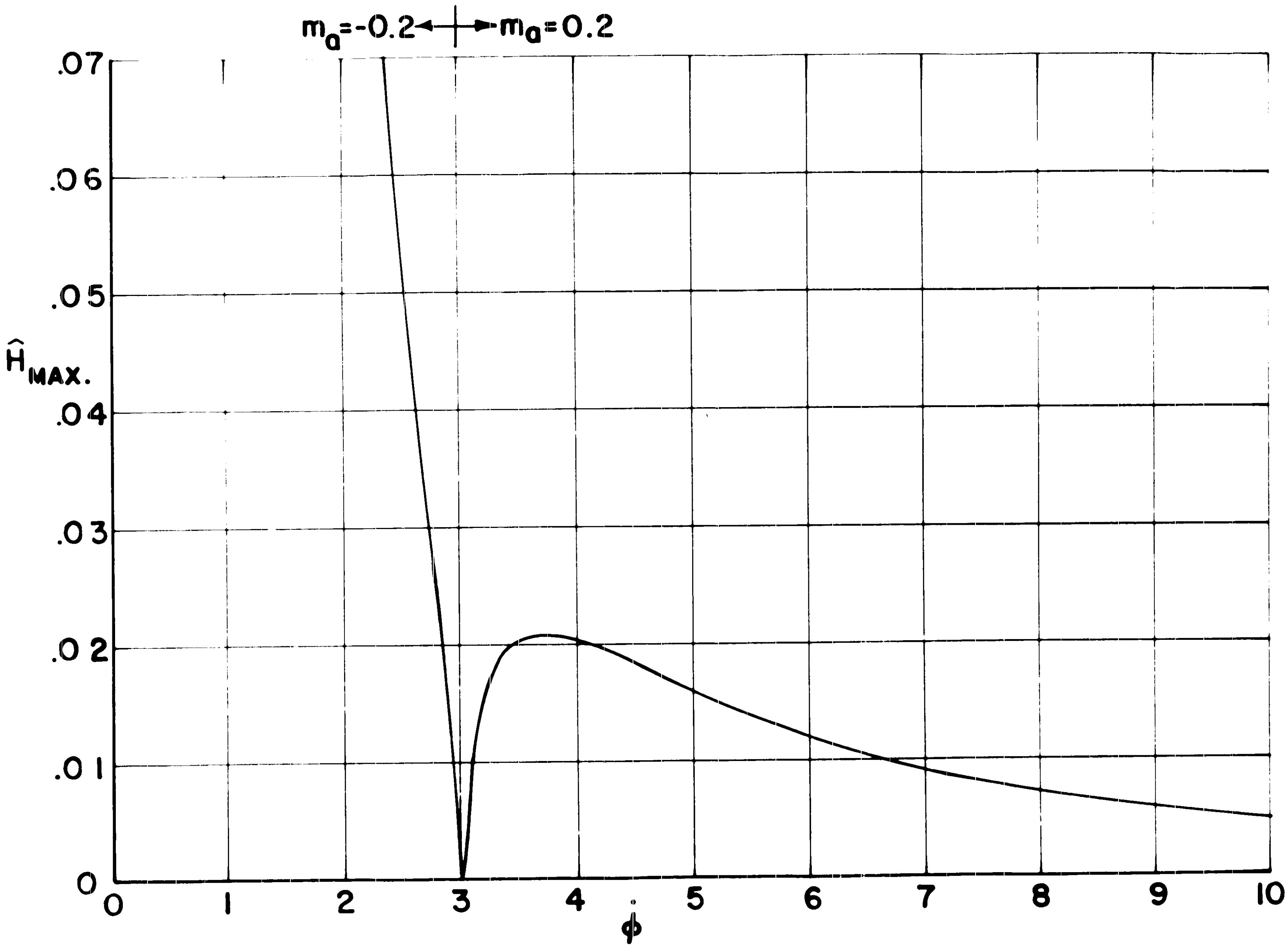


Figure 4. Maximum Damping Rate as a Function of Spin for $m_a = \pm 0.2$

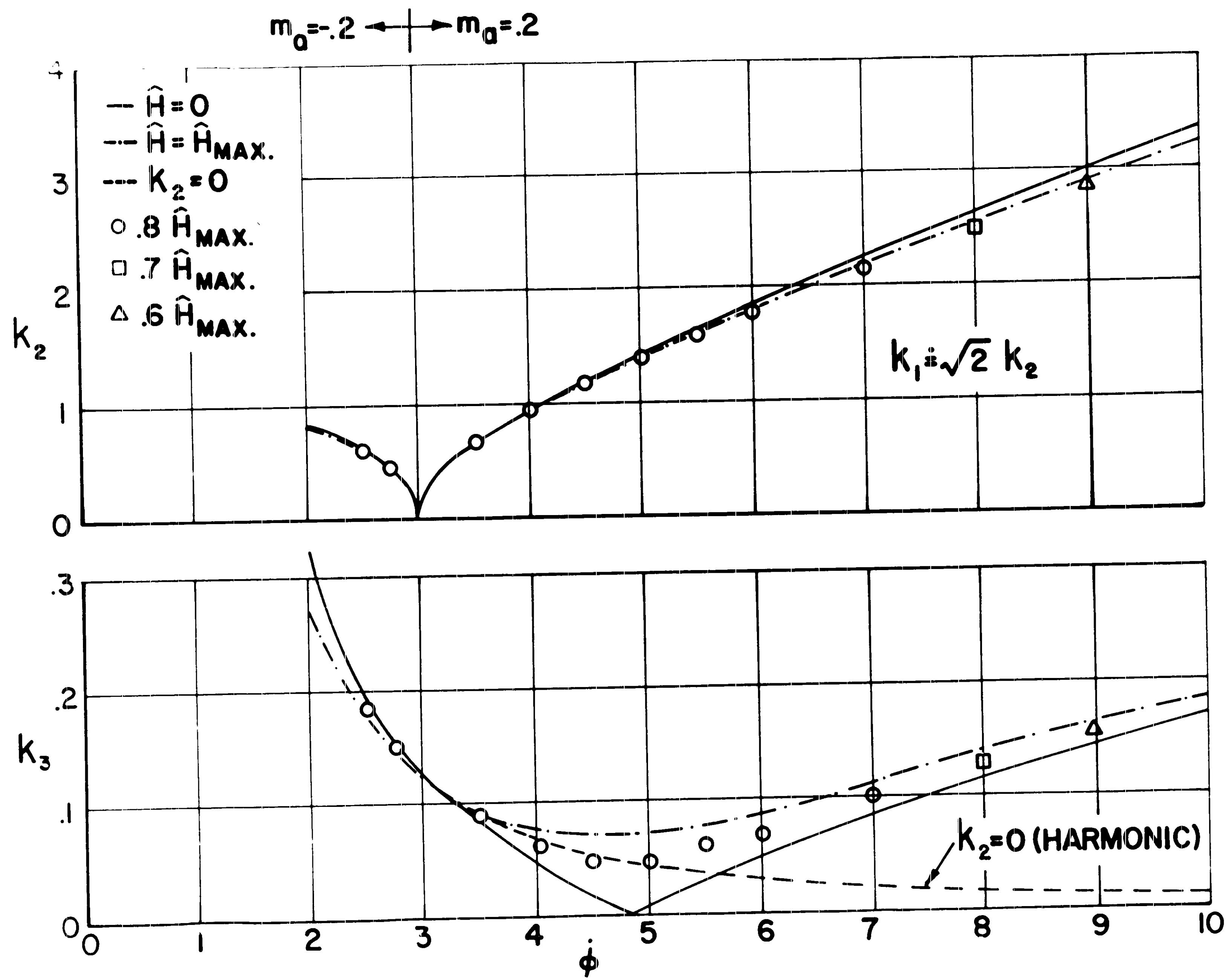


Figure 5. $k_2$ and $k_3$ versus Spin for $m_a = \pm\ 0.2$ and for Both Maximum and Zero Aerodynamic Damping

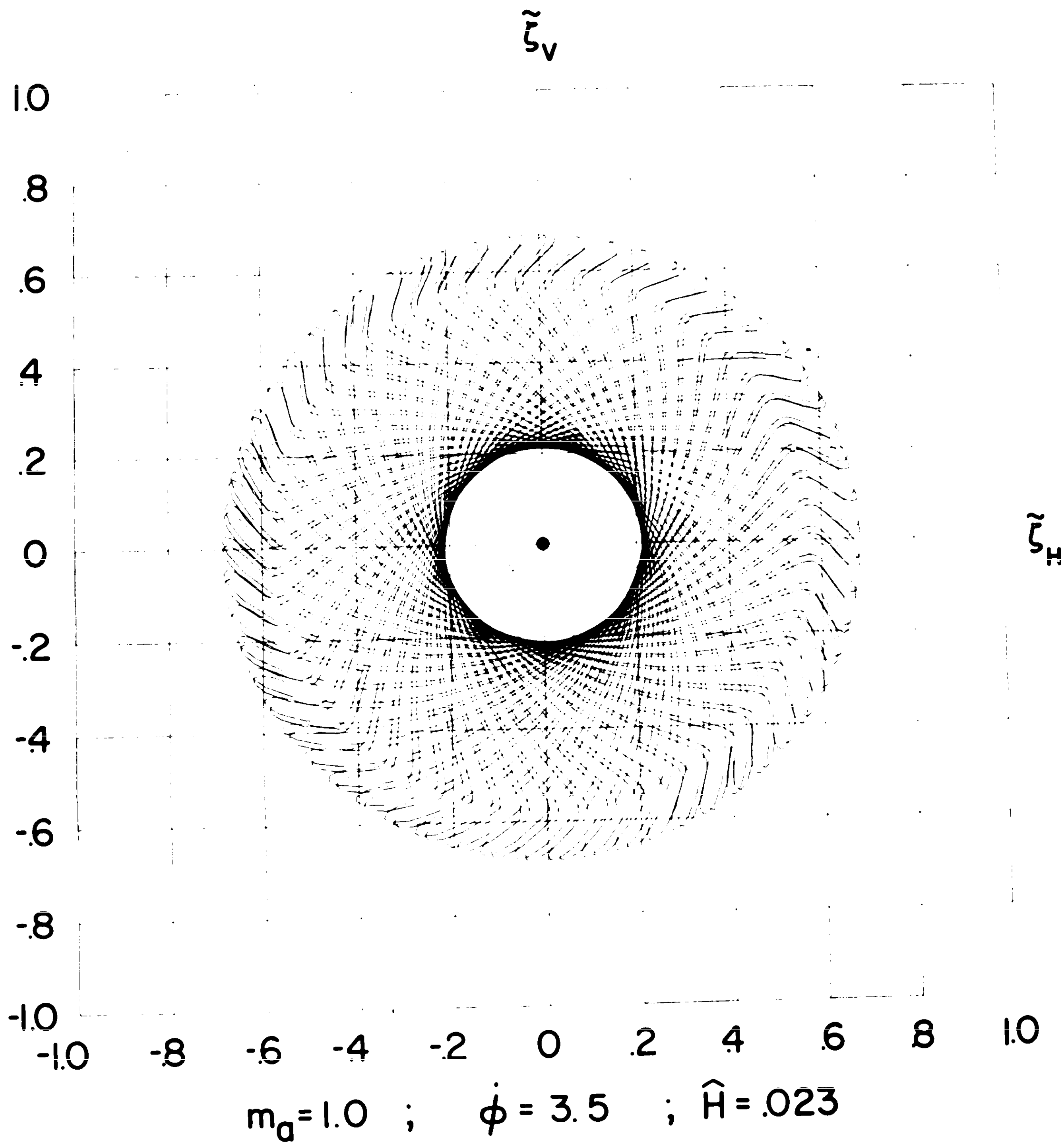


Figure 6. Angular Motion for Parameters of Table II in Nonspinning Coordinates

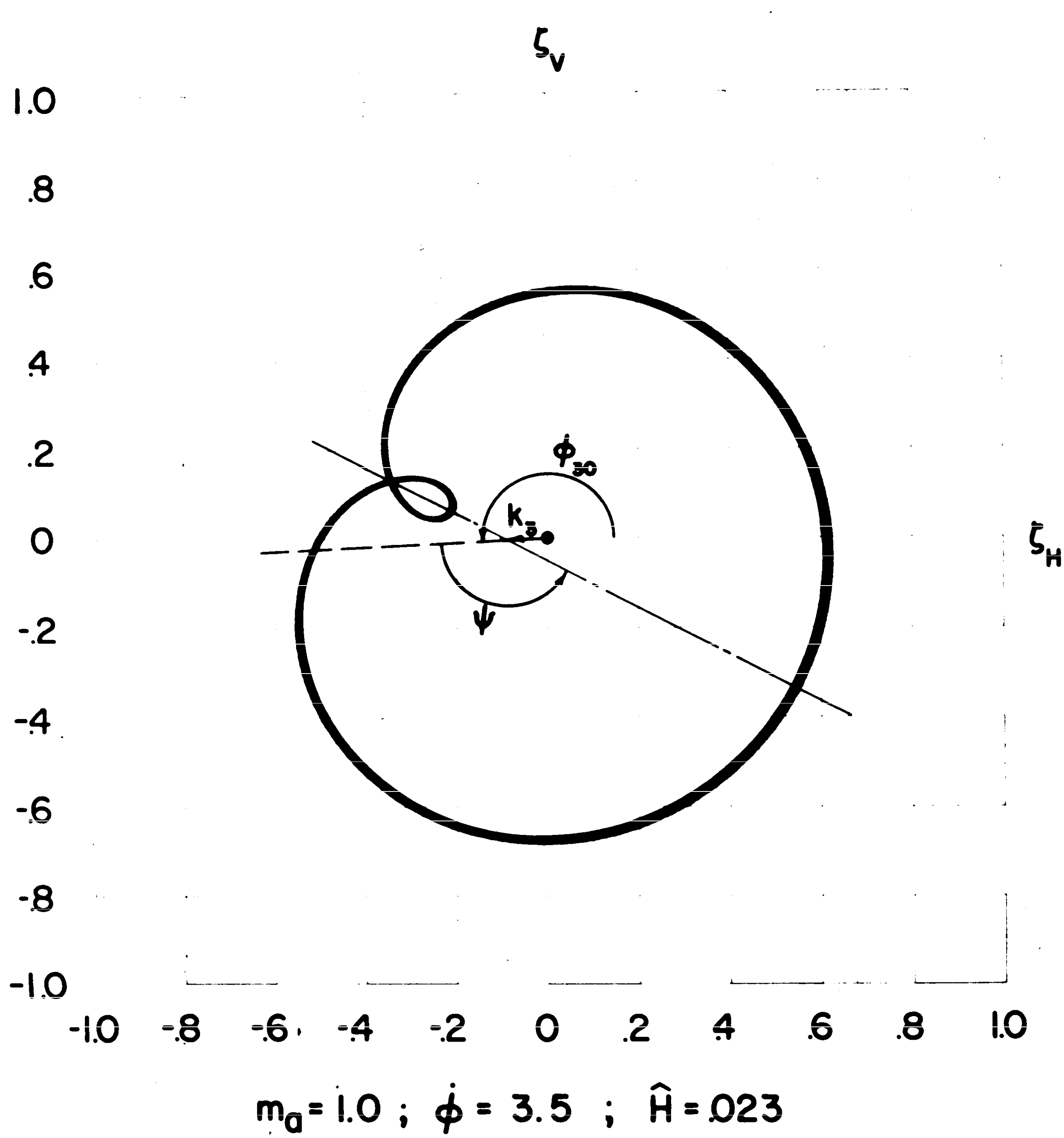


Figure 7. Angular Motion for Parameters of Table II in Missile Fixed Coordinates

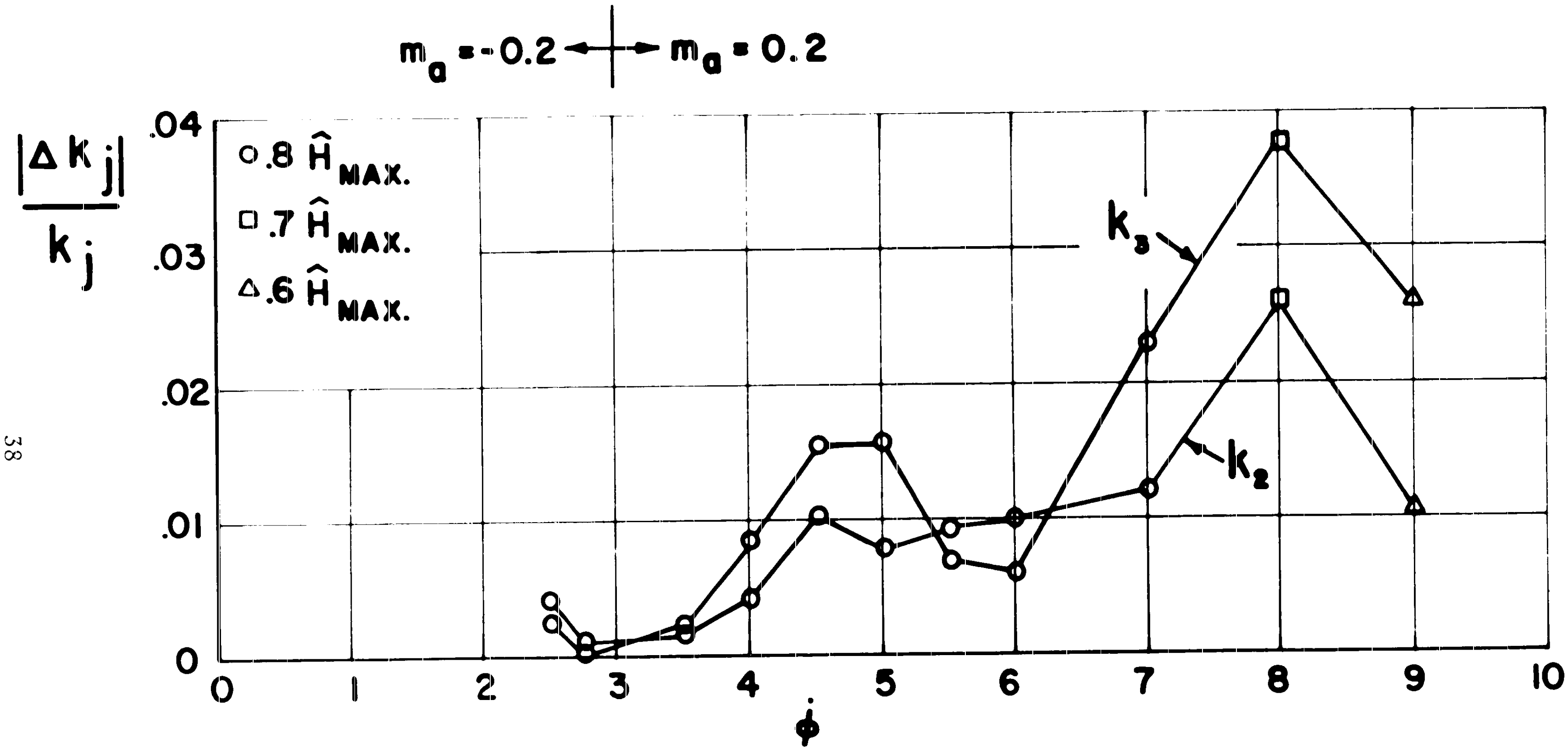


Figure 8. Difference Between Quasilinear Predictions of $k_2$ and $k_3$ and Results of Numerical Integration of Exact Differential Equations

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## APPENDIX A

## DERIVATION OF EQUATION (25)

If we assume $k_3 = 0$ and $b = 1$, the frequency equations of Table I become

$$\dot{\phi}_1 = [1 + 4\ m_a\ k_2^2]^{\frac{1}{2}} \tag{A-1}$$

$$\dot{\phi}_2 = -\ [1 + 5\ m_a\ k_2^2]^{\frac{1}{2}} \tag{A-2}$$

Substituting Equations (A-1--A-2) in Equation (21), we have

$$\dot{\phi} = 2[1 + 4\ m_a\ k_2^2]^{\frac{1}{2}} + [1 + 5\ m_a\ k_2^2]^{\frac{1}{2}} \tag{A-3}$$

Equation (A-3) can now be simplified by squaring it twice.

$$121\ m_a^2\ k_2^4 + 6(11 - 7\ \dot{\phi}^2)\ m_a\ k_2^2 + (\dot{\phi}^2 - 9)(\dot{\phi}^2 - 1) = 0 \tag{A-4}$$

$$\therefore\ k_2^2 = \frac{-\ 3(11 - 7\ \dot{\phi}^2) \pm [9(11 - 7\ \dot{\phi}^2)^2 - 121(\dot{\phi}^2 - 9)(\dot{\phi}^2 - 1)]^{\frac{1}{2}}}{121} \tag{A-5}$$

According to Equation (A-3), $\dot{\phi}$ is 3 when $k_2$ vanishes. Thus the root with the minus sign in Equation (A-5) is extraneous. We therefore take the root with the plus sign and approximate the radical with the first two terms of the binomial expansion.

$$\therefore m_a k_2^2 \doteq \frac{(\dot{\phi}^2 - 9)(\dot{\phi}^2 - 1)}{6(7\dot{\phi}^2 - 11)} \quad . \tag{A-6}$$

Near $\dot{\phi} = 3$, we can approximate $\dot{\phi}$ with 3 in every factor of Equation (A-6) except $(\dot{\phi} - 3)$.

$$\therefore m_a k_2^2 = 2(\dot{\phi} - 3)/13 \tag{A-7}$$

## APPENDIX B

## EFFECT OF LIFT, DRAG AND MAGNUS MOMENT

A linear Magnus moment can be easily introduced into Equation (3)

$$C_{\tilde m} + i\, C_{\tilde n} = -\, i\, C_{M_0}\, e^{i\phi} - i(c_0 + c_2\delta^2)\tilde{\xi}$$

$$-\, i\, C_{M_{\dot\alpha}}\, \tilde{\xi}' + C_{M_q}\, \tilde{\mu} \qquad \text{(B-1)}$$

$$+\left(\frac{p\ell}{V}\right) C_{M_{p\alpha}}\, \tilde{\xi}$$

With this linear Magnus moment, a linear lift force and constant drag coefficient, Equation (4) becomes:[11]

$$\tilde{\xi}'' + (H - iP)\tilde{\xi}' - [(M_0 + M_0\, \delta^2) + iPT)\tilde{\xi} = -\, M_a\, e^{i\phi} \qquad \text{(B-2)}$$

$$\text{where } H = \frac{\rho S \ell^3}{2\, I_y}\left[\frac{I_y}{m\ell^2}\left(C_{L_\alpha} - C_D\right) - \left(C_{M_q} + C_{M_{\dot\alpha}}\right)\right]$$

$$T = \frac{\rho S \ell}{2\, I_x}\left[\frac{I_x}{m\ell^2}\, C_{L_\alpha} + C_{M_{p\alpha}}\right]$$

and the other symbols are unchanged.

The changes of dependent and independent variables yield

$$\ddot{\tilde{\xi}} + (\hat{H} - i\hat{P})\dot{\tilde{\xi}} + (1 + m_a |\tilde{\xi}|^2 + i \, P\hat{T})\hat{\xi} = e^{i\phi} \qquad \text{(B-3)}$$

The harmonic response solution to Equation (B-3) is quite similar to Equation (11):

$$k_3 e^{i\phi_{30}} = [1 - (1 - I_x/I_y)\dot{\phi}^2 + m_a k_3^2 + i \, \dot{\phi}(\hat{H} - I_x\hat{T}/I_y)]^{-1} \qquad \text{(B-4)}$$

The presence of $\hat{T}$ affects the quasilinear analysis through the damping equations (Equations (16-17)) and one of the harmonic component relations (Equation (20)). For constant frequencies and constant $\Psi$, the equations for the damping exponents become:[10]

$$\hat{\lambda}_1 = - [\hat{H}\dot{\phi}_1 - \hat{P}\hat{T} - 2 \, m_a \, k_2 \, k_3 \, \sin\Psi][2\dot{\phi}_1 - \hat{P}]^{-1} \qquad \text{(B-5)}$$

$$\hat{\lambda}_2 = - [H\dot{\phi}_2 - \hat{P}\hat{T} + m_a \, k_1^2 \, k_2^{-1} \, k_3 \, \sin\Psi][2\dot{\phi}_2 - \hat{P}]^{-1} \qquad \text{(B-6)}$$

The revised version of Equation (20) is:

$$k_3 \, (\dot{\phi}\hat{H} - \hat{P}\hat{T}) + m_a \, k_1^2 \, k_2 \sin\Psi = - \sin \phi_{30} \qquad \text{(B-7)}$$

We now require $\hat{\lambda}_j = 0$ and obtain revised versions of Equations (22-23):

$$\sin\Psi = (H\dot{\phi}_1 - \hat{P}\hat{T})(2\, m_a\, k_2\, k_3)^{-1} \tag{B-8}$$

$$k_1^2 = 2\, b\, k_2^2 \tag{B-9}$$

where $b = -(H\dot{\phi}_2 - \hat{P}\hat{T})/(H\dot{\phi}_1 - \hat{P}\hat{T})$

Equations (18-19, 21, B-7--B-9) once again provide a set of seven equations in the seven unknowns $(k_1, k_2, k_3, \dot{\phi}_1, \dot{\phi}_2, \phi_{30}, \Psi)$.

## APPENDIX C

## STABILITY OF HARMONIC MOTION

The harmonic solution to Equation (8), given by Equation (11), can be multivalued. It is shown in Reference 7 that the intermediate size solution is unstable and this solution is, therefore, identified by a dashed line in Figure 1. The stability argument is quite simple and will be repeated here for the convenience of the reader. The similar stability argument for Duffing's Equation[13] is much more complicated and involves a detailed discussion of Mathieu's Equation.

We first perturb the harmonic solution of Equation (8) by a small complex function $\eta$ .

$$\tilde{\xi} = (k_3 + \eta)e^{i\phi_3} \tag{C-1}$$

Equation (8) then takes on the form

$$\ddot{\eta} + [\hat{H} + i(2 - I_x/I_y)\dot{\phi}]\,\dot{\eta} + [1 - (1 - I_x/I_y)\dot{\phi}^2 + 2\,m_a\,k_3^2 + i\hat{H}\dot{\phi}]\eta$$
$$+ m_a\,k_3^2\,\bar{\eta} = 0 \tag{C-2}$$

if all higher order terms in $\eta$ and its derivatives are neglected.

Next we assume an exponential solution for $\eta$ and separate into real and imaginary parts

$$\eta = (\eta_{10} + i\ \eta_{20})e^{\lambda\tau} \tag{C-3}$$

$$\{\lambda^2 + \hat{H}\lambda + [1 - (1 - I_x/I_y)\dot{\phi}^2 + 3\ m_a\ k_3^2]\}\eta_{10}$$

$$- [(2 - I_x/I_y)\dot{\phi}\lambda + \hat{H}\dot{\phi}]\eta_{20} = 0 \tag{C-4}$$

$$[(2 - I_x/I_y)\dot{\phi}\lambda + \hat{H}\dot{\phi}]\eta_{10}$$

$$+ \{\lambda^2 + \hat{H}\lambda + [1 - (1 - I_x/I_y)\dot{\phi}^2 + m_a\ k_3^2]\}\eta_{20} = 0 \tag{C-5}$$

The characteristic equation for $\lambda$ can now be obtained by setting the determinant of the coefficients of $\eta_{j0}$ in Equations (C-4) and (C-5) to zero. For simplicity we will neglect the ratio of the moments of inertia in comparison to one.

$$\lambda^4 + a\ \lambda^3 + b\ \lambda^2 + c\ \lambda + d = 0 \tag{C-6}$$

where

$$a = 2\hat{H}$$

$$b = 2(1 + \dot{\phi}^2) + 4\, m_a\, k_3^2 + \hat{H}^2$$

$$c = 2\hat{H}[1 + \dot{\phi}^2 + 2\, m_a\, k_3^2]$$

$$d = (1 - \dot{\phi}^2 + 3\, m_a\, k_3^2)(1 - \dot{\phi}^2 + m_a\, k_3^2) + \hat{H}^2\phi^2$$

The usual analysis can now be employed on the coefficients of this quartic equation to determine the existence of a positive real part of $\lambda$ . For small damping, this analysis can be considerably simplified since a and c can then be neglected and Equation (C-6) becomes a quadratic equation in $\lambda^2$ with solution

$$\lambda^2 = -\, b/2 \pm \sqrt{(b/2)^2 - d} \qquad \text{(C-7)}$$

For positive $m_a$, b is always positive and $\lambda$ will have a positive real part only when

$$d = (1 - \dot{\phi}^2 + 3\, m_a\, k_3^2)(1 - \dot{\phi}^2 + m_a\, k_3^2) < 0 \qquad \text{(C-8)}$$

For negative $m_a$, the situation is more complicated since b can change sign. The analysis can, of course, be performed but will not be given here.

Under our approximation, Equation (11) for the harmonic amplitude and phase becomes

$$k_3 \, [1 - \dot{\phi}^2 + m_a \, k_3^2] = \pm 1 \qquad \text{(C-9)}$$

Thus the curve

$$1 - \dot{\phi}^2 + m_a \, k_3^2 = 0 \qquad \text{(C-10)}$$

is the asymptote of $k_3$. As can be seen in Figure C-1, it also can be called the "backbone" of the harmonic response curve.

If we differentiate Equation (C-9), we have

$$k_3 \, \dot{\phi} \, \frac{d\dot{\phi}}{dk_3} = 1 - \dot{\phi}^2 + 3 \, m_a \, k_3^2 \qquad \text{(C-11)}$$

Thus the first term in d vanishes when $d\dot{\phi}/dk_3$ is zero. The curve

$$1 - \dot{\phi}^2 + 3 \, m_a \, k_3^2 = 0 \qquad \text{(C-12)}$$

is also shown in Figure 9. d is negative between these curves and the intermediate size solution for $k_3$ is, therefore, unstable.

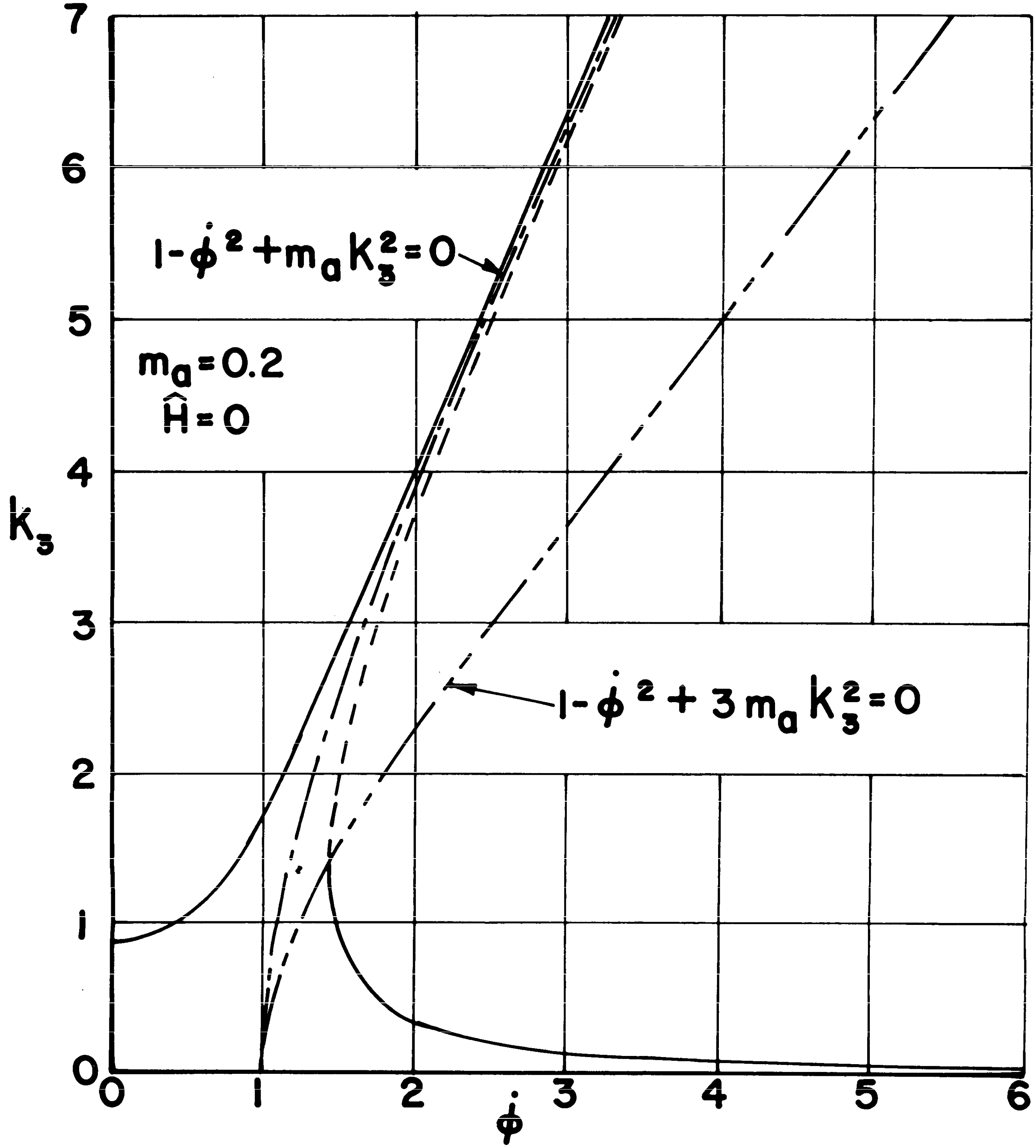


Figure C-1. Stability Boundaries for Harmonic Response

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13. ABSTRACT

The quasilinear analysis of the angular motion of slightly asymmetric missiles with a cubic static moment shows the existence of nonharmonic steady-state solutions. These solutions are described by sums of three constant amplitude rotating angles. One angle rotates at the spin rate and the other two rotate in opposite directions at approximately one-third the spin rate. This generalized subharmonic motion can be very much larger than the usual harmonic trim motion and can occur for spin rates greater than eight times the resonant spin rate. The predictions of the quasilinear theory are compared with the results of exact numerical integration of the equations of motion with much better than 5% agreement in all cases.

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| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Spin-Yaw Resonance | | | | | | |
| Asymmetric Missiles | | | | | | |
| Varying Roll Trim | | | | | | |
| Subharmonic Resonance | | | | | | |

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